

EULER EQUATIONS AND TURBULENCE: ANALYTICAL APPROACH TO INTERMITTENCY

A. CHESKIDOV AND R. SHVYDKOY

ABSTRACT. In this note we introduce in precise mathematical terms some of the empirical concepts used to describe intermittency in a fully developed turbulence. We will give definitions of the active turbulent region, volume, eddies, energy dissipation set, and derive rigorously some power laws of turbulence. In particular, the formula for the Hausdorff dimension of the energy dissipation set will be justified, and upper/lower bounds on the energy spectrum will be obtained.

1. INTRODUCTION

Intermittency, if viewed as a measure of non-uniformity of the energy cascade, bears its signature on many statistical laws of a fully developed turbulence. A typical example is the power law for the energy density function,

$$(1) \quad E(\kappa) \sim \frac{\varepsilon^{\frac{2}{3}}}{\kappa^{\frac{5}{3}}} \left(\frac{\kappa_0}{\kappa} \right)^{1-\frac{d}{3}},$$

where ε is the energy dissipation rate per unit mass, κ_0 the integral scale, and d is the dimension of a set that carries turbulent energy dissipation. One observes similar corrections for longitudinal structure functions or skewness and flatness factors of velocity gradients (see the texts [15, 18, 17] for detailed discussions). Due to intermittency the deterministic counterparts of the power laws appear in a form of bounds to account for possible corrections as in Constantin et al [9, 6, 7], even though the intermittency itself has not been mathematically defined. As Foias remarked [13], “The rigorous mathematical reformulation of the heuristic estimates provided by some widely accepted turbulence theories always seem to represent the upper bound of the rigorous mathematical estimates.” At present there is no compelling analytical evidence that the fluid equations are consistent with the full range

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of intermittent dimensions $0 \leq d \leq 3$ (or with $d < 0$). On the other hand, experimental evidence points only to moderate deviations from the classical Kolmogorov theory. Based on several subjective factors, such as the overwhelming success of widely used hot wire anemometry, or high incidence rates of linear cross-sections with (2+)D fractals (see [16]), one is lead to believe that the values of d above 2 are more likely to occur in real turbulence. For Leray-Hopf solutions of the Navier-Stokes equation this question is directly related to the regularity problem for the following reason. If d is defined as a saturation level of Bernstein's inequality at the upper end of the inertial range (in a proper sense) then, as shown in [3], a solution with dimension d above $3/2$ is automatically regular. We note that such an interpretation of intermittency also appeared in a derivation of the dyadic [4] and continuous [5] models of the energy cascade.

Without implementation of viscous dissipation the statistical theory offers examples of models where the dimension is introduced as a rarefaction exponent of Richardson's cascade (see [14, 19, 16]). Let us briefly recall the β -model of Frisch, Sulem, and Nelkin [14]. Assume that a fluid is in a state of turbulent motion confined to the periodic box $\mathbb{T}_L^3 = [0, L]^3$ and observed on a long interval of time $[0, T]$. On each stage of the cascade the kinetic energy is carried through by a set of active eddies $\{\epsilon_q\}$. Let the eddies fill a volume V_q . If one assumes that V_q is a fraction of the preceding volume, i.e. $V_q \sim \beta V_{q-1}$, for some $0 < \beta \leq 1$, then going up the scales, $V_q \sim \beta^q L^3$. Letting $\mathfrak{h} = -\log_2 \beta$, we have $V_q \sim 2^{-q\mathfrak{h}} L^3$ and $\mathfrak{h} \geq 0$. A heuristic incidence argument shows that the ultimate accumulation of the cascade occurs on a set that crosses infinitely many generations of eddies, i.e. $A = \limsup_{q \rightarrow \infty} \cup \epsilon_q$, and the dimension of the accumulant A does not exceed $d = 3 - \mathfrak{h}$. A simple dimensional analysis recovers the energy spectrum law (1). Further hypothesis of a statistical uni-scaling of velocity displacements $\delta u(\ell)$ on A made the model unsuitable to capture experimentally observed exponents for higher order structure functions. This lead to subsequent multi-fractal ramifications of Frisch and Parisi (see [21, 12, 15]).

Our objective in this paper is to define the above mentioned empirical concepts for deterministic fields, and use them to justify intermittency corrections to statistical laws without the use of viscous dissipation. We outline our basic definitions and results. Let $u \in L^2([0, T] \times \mathbb{T}_L^3)$ be a divergence free time dependent vector field. We will use the Littlewood-Paley decomposition of u given by $u = \sum_{q \geq -1} u_q$ (see Section 2.1). We interpret u_q as the collective velocity field of all eddies ϵ_q of dyadic size

$\ell_q = L/2^q$. By bridging relations between the Eulerian and Lagrangian description of the energy flux through scales ℓ_q we define the active volume as

$$(2) \quad V_q = L^3 \frac{\langle |u_q|^2 \rangle^3}{\langle |u_q|^3 \rangle^2}.$$

Here and throughout the bracket $\langle \cdot \rangle$ denotes the average over the space-time domain $\Omega_T = [0, T] \times \mathbb{T}_L^3$. The co-dimension \mathfrak{h} is then introduced as the following exponential type of the sequence $\{V_q\}$:

$$(3) \quad \mathfrak{h} = \liminf_{q \rightarrow \infty} \frac{\log_2(L^3/V_q)}{q}.$$

Note that \mathfrak{h} or the dimension $d = 3 - \mathfrak{h}$ is a number "attached" to the field u without any assumptions on u . By the Hölder inequality we have one obvious bound $V_q \leq L^3$, and in the stationary case Bernstein's inequality implies $V_q \geq c\ell_q^3$, where c is a-dimensional. The latter implies $d \geq 0$, which is a natural bound. However, in the time-dependent case, V_q can scale down faster due to possible temporal rarefaction of active regions giving negative values of the dimension d (see also a discussion in Frisch [15]).

To define active regions that fill volume V_q we find that for an eddy to qualify as "active" it has to have speed at least proportional to

$$s_q \sim \frac{\langle |u_q|^3 \rangle}{\langle |u_q|^2 \rangle}.$$

The speeds will be interpreted as magnitudes of coefficients in the atomic decomposition of u_q , and the supports of atoms will be viewed as eddies. The collection of those eddies will constitute what we call an "active" region A_q . The number of active eddies is represented by V_q/ℓ_q^3 . Our Theorem 3.2 shows that the Hausdorff dimension of

$$A = \limsup_{q \rightarrow \infty} A_q$$

is not to exceed d as predicted by the model, and A_q supports most of the "active" part of the field u_q in terms of L^3 -averages. We will recover the energy law (1) in the form of an upper and lower bounds. The scaling of the second order structure function is as predicted by the β -model

$$(4) \quad S_2(\ell) \lesssim \varepsilon^{\frac{2}{3}} \ell^{\frac{2}{3}} \left[\left(\frac{\ell}{L} \right)^{1 - \frac{d}{3} - \delta} + C_\delta \left(\frac{\ell}{L} \right)^{\frac{4}{3}} \right].$$

for all $\ell < L$, and any $\delta > 0$. Here and throughout, \lesssim denotes inequality up to a universal adimensional constant.

In Section 4 we stipulate on the role of A as an accumulator of the energy cascade for weak solutions to the Euler equations. We define the energy flux through scales ℓ_q by $\Pi_q = \langle \pi_q \rangle$ where density π_q is the trilinear term that contains all relevant interactions involved in the energy transfer. Due to a localization of Π_q observed in [2] all appreciable interactions are frequency-local near the ℓ_q scales. We show in Theorem 4.3 that the complement of A takes passive part in the cascade process in the sense that there is a nested sequence of sets $G_p \rightarrow A$, in fact $G_p = \cup_{k>p} A_k$, such that for all $p > 0$

$$\lim_{q \rightarrow \infty} \langle |\pi_q|_{\Omega_T \setminus G_p} \rangle = 0.$$

In other words, the turbulent cascade tends to dump the energy on the carrier A . The results of this section are established under the natural Onsager regularity condition $\varepsilon = \limsup_{q \rightarrow \infty} \langle |u_q|^3 \rangle / \ell_q < \infty$ which is known to be suitable for the turbulent interpretation of the field (see [2, 8, 23, 10, 20, 11]). Any better regularity of u , namely simply requiring $\varepsilon = 0$, erases turbulent energy dissipation. This lack of dissipation may also occur locally in a proper sense. Therefore it is necessary to further confine the cascade to the smaller set $A \cap S$, where S is the Onsager singular set of u , i.e. S is the smallest set on the complement of which $\varepsilon = 0$. This reduction is performed in Section 4.3. We will demonstrate by several examples that the regularity-related intermittency due to S and the cascade-related intermittency due to A are essentially independent. Therefore one has to consider both to have a more complete description of the accumulator.

The active regions A_q can be used to give a more detailed description of a measure-theoretic support of the anomalous energy dissipation introduced by Duchon and Robert in [10]. Recall that for every weak solution $u \in L^3(\Omega_T)$ to the Euler equation one can associate a distribution $\mathcal{D}(u)$ such that

$$(5) \quad \partial_t \left(\frac{1}{2} |u|^2 \right) + \operatorname{div} \left(u \left(\frac{1}{2} |u|^2 + p \right) \right) + \mathcal{D}(u) = 0.$$

In our notation, and under the assumption $\varepsilon < \infty$, $\mathcal{D}(u)$ is obtained as a weak limit of the sequence $\{\pi_q\}_q$ which is uniformly bounded in $L^1(\Omega_T)$. Thus, $\mathcal{D}(u)$ is a measure of bounded variation. One can easily show that $\operatorname{supp} \mathcal{D}(u) \subset S$, but it is unclear whether the topological support of $\mathcal{D}(u)$ is contained in the accumulator A or is at all relevant to the intermittent cascade. Instead, we will show that the measure-theoretic support of $\mathcal{D}(u)$, in the sense of Hahn decomposition, can in fact be described by limits of algebraic progressions of the active regions A_q in a sense defined in Section 6. However, due to a possible

slow convergence rate of such a limit, $\mathcal{D}(u)$ may not enjoy the same dimensional bound as A itself.

2. PHENOMENOLOGY OF ACTIVE VOLUMES

2.1. Dimensional Fourier analysis on a periodic box. Let $B(0, r)$ denote the ball centered at 0 of radius r in \mathbb{R}^3 . We fix a nonnegative radial function χ belonging to $C_0^\infty(B(0, 1))$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1/2$. We define $\phi(\xi) = \chi(\xi/2) - \chi(\xi)$ and let

$$h(x) = \int_{\mathbb{R}^3} \phi(\xi) e^{2\pi i \xi \cdot x} dx,$$

the usual inverse Fourier transform of ϕ on \mathbb{R}^3 . We now introduce a dimensional dyadic decomposition. Let $L > 0$ be fixed length scale and let $\varphi_{-1}(\xi) = \chi(L\xi)$ and $\varphi_q(\xi) = \phi(L\xi/2^q)$ for $q \geq 0$. We have $\sum_{q \geq -1} \varphi_q(\xi) = 1$, and

$$(6) \quad \text{supp } \varphi_q \cap \text{supp } \varphi_p = \emptyset \quad \text{for all } |p - q| \geq 2.$$

Let \mathbb{T}_L^3 denote the torus of side length L . Let us denote by \mathcal{F} the Fourier transform on \mathbb{T}_L^3 :

$$\mathcal{F}(u)(k) = \frac{1}{L^3} \int_{\mathbb{T}_L^3} u(x) e^{-2\pi i k \cdot x} dx, \quad k \in \frac{1}{L} \mathbb{Z}^3,$$

then

$$\mathcal{F}^{-1}(f)(x) = \sum_{k \in \frac{1}{L} \mathbb{Z}^3} f(k) e^{2\pi i k \cdot x}$$

determines the inverse transform. We define

$$u_q = \mathcal{F}^{-1}(\varphi_q \mathcal{F}(u)).$$

Here φ_q must be understood as restricted on the lattice $\frac{1}{L} \mathbb{Z}^3$. In particular, $u_{-1}(x) = \frac{1}{L^3} \int_{\mathbb{T}_L^3} u(y) dy$. It is informative to note the formula for $h_q = \mathcal{F}^{-1} \varphi_q$, $q \geq 0$, which can be derived via the standard transference argument:

$$(7) \quad h_q(x) = 2^{3q} \sum_{n \in \mathbb{Z}^3} h\left(\frac{2^q x}{L} + 2^q n\right), \quad x \in \mathbb{T}_L^3.$$

Then for all $r \in [1, \infty]$ we have

$$(8) \quad \|h_q\|_r = \left(\frac{1}{L^3} \int_{\mathbb{T}_L^3} |h_q(x)|^r dx \right)^{1/r} \sim 2^{3q(r-1)/r}$$

which is independent of the scale. From (8) follow the classical Bernstein's inequalities, which are also scaling invariant:

$$(9) \quad \|u_q\|_{r''} \leq c 2^{3q(\frac{1}{r'} - \frac{1}{r''})} \|u_q\|_{r'}$$

for all $1 \leq r' < r'' \leq \infty$; and the differential Bernsteins's inequalities:

$$(10) \quad \|\partial^\beta u_q\|_r \sim (2^q/L)^{|\beta|} \|u_q\|_r, \quad q \geq 0.$$

We will use the special notation $\lambda_q = 2^q/L$ for the dimensional wave-numbers in order to distinguish them from the a-dimensional ones 2^q . The corresponding dyadic length scale is denoted $\ell_q = \lambda_q^{-1}$.

2.2. Active volumes. We assume that a turbulent fluid fills a periodic box of linear dimension L . Let us denote it by \mathbb{T}_L^3 . The motion of the fluid is driven by an external stirring force f , which we assume to be time independent and of scale $\eta_f \sim L$. More specifically, $\text{supp } \mathcal{F}(f) \subset B(0, c/L)$ for some a-dimensional $c > 1$. We observe the fluid on a time interval $[0, T]$ long enough to capture the necessary statistics. Let us envision that the turbulent motion of the fluid at scale λ_q consists of actively interacting eddies and these eddies fill a region of volume V_q , which we also call active. Let us now proceed using the classical phenomenological argument, giving every concept its Littlewood-Paley analogue. Our immediate goal will be to derive the following explicit formula for V_q :

$$(11) \quad V_q = L^3 \frac{\langle |u_q|^2 \rangle^3}{\langle |u_q|^3 \rangle^2}.$$

Here the bracket $\langle \cdot \rangle$ denote the average in space-time on the domain $\Omega_T = [0, T] \times \mathbb{T}_L^3$. Let us start by noticing that since φ_q has mean zero, for $q \geq 0$, we can prescribe the turn-over velocity of an ℓ_q -eddy at location x to be $u_q(x)$. Let U_q be a characteristic velocity of an ℓ_q -eddy. In order to single out active eddies from passive ones, one will use an L^p -average such as $U_q \sim \langle |u_q|^p \rangle^{1/p}$. The minimal value of p proved to be suitable for studying intermittent cascade turns out to be 3 (see Section 4), while higher values can be adopted to formulate multi-fractal hypotheses as shown in Section 4.5. So, we define

$$(12) \quad U_q = \frac{L}{V_q^{1/3}} \langle |u_q|^3 \rangle^{1/3},$$

or more explicitly,

$$(13) \quad U_q = \left(\frac{1}{TV_q} \int_{\Omega_T} |u_q(x, t)|^3 dx dt \right)^{1/3},$$

Even though the integration in (13) is performed over the entire domain, we will prove later that there exists indeed an active region $A_q \subset \Omega_T$ with $|A_q| \lesssim TV_q$, and

$$(14) \quad \int_{A_q} |u_q|^3 dx dt \sim \int_{\Omega_T} |u_q|^3 dx dt.$$

The input energy produced by the force f is passing from larger to smaller scales. The energy flux per unit volume carried through the scales of order ℓ_q is given by

$$(15) \quad \varepsilon_q = \frac{U_q^3}{\ell_q} = \frac{L^3}{\ell_q V_q} \langle |u_q|^3 \rangle.$$

On the other hand,

$$(16) \quad \varepsilon_q = \frac{K_q}{t_q}$$

where K_q is the average kinetic energy of an active ℓ_q -eddy given by

$$(17) \quad K_q = \frac{L^3}{V_q} \langle |u_q|^2 \rangle,$$

(here again we use the active proportion of the volume) and t_q is the typical turnover time given by

$$(18) \quad t_q = \frac{\ell_q}{U_q} = \frac{\ell_q V_q^{1/3}}{L \langle |u_q|^3 \rangle^{1/3}}.$$

Putting together (17) and (18) we obtain another expression for ε_q :

$$(19) \quad \varepsilon_q = \frac{L^4}{\ell_q V_q^{4/3}} \langle |u_q|^2 \rangle \langle |u_q|^3 \rangle^{1/3}.$$

Equating (15) and (19) we finally obtain (11).

2.3. Intermittent dimension. We define the co-dimension \mathfrak{h} as an asymptotic scaling exponent:

$$(20) \quad \mathfrak{h} = \liminf_{q \rightarrow \infty} \frac{\log_2(L^3/V_q)}{q}.$$

Let the dimension be $d = 3 - \mathfrak{h}$. Observe that since all volumes are bounded by L^3 , we have $d \leq 3$. The dimension d is defined without any scaling hypothesis like in the β -model and therefore is a property of the field.

3. ACTIVE REGIONS

Let us assume as before that $u(x, t)$ is the velocity field of a turbulent fluid in the periodic box \mathbb{T}_L^3 on a time interval $[0, T]$, and V_q 's are defined by (11). In this section we will introduce regions that capture most of the active eddies, i.e. eddies with at least a certain turn-over speed s_q . In order to properly define the speed and the regions we will make use of atomic decompositions. Let us briefly recall the definition (for more, see [1]). For $q \geq 0$ and $k \in [0, 2^q - 1]^3 \cap \mathbb{Z}^3$ we define the dyadic cubes

$$Q_{qk} = [0, \ell_q)^3 + \ell_q k \subset \mathbb{T}_L^3,$$

and dilated cubes

$$Q_{qk}^* = \bigcup_{|k' - k| < 3} Q_{qk'}.$$

For any $M > 1$ one can find a spacial decomposition of $u(t, \cdot)$ into M -atoms

$$(21) \quad u(t, x) = \sum_{q=0}^{\infty} \sum_{k \in [0, 2^q - 1]^3 \cap \mathbb{Z}^3} s_{qk}(t) a_{qk}(t, x),$$

where $\text{supp}(a_{qk}(t, \cdot)) \subset Q_{qk}^*$ compactly, and

$$\max_{|\beta| \leq M} \{ \lambda_q^{-|\beta|} \|\partial_x^\beta a_{qk}\|_{L^\infty(\Omega_T)} \} \leq 1.$$

For every $r < \infty$ one has

$$(22) \quad \left(\frac{1}{\lambda_q^3} \sum_{k \in [0, 2^q - 1]^3} |s_{qk}(t)|^r \right)^{1/r} \sim \|u_q(t, \cdot)\|_r.$$

Let us recall how the coefficients s_{qk} are constructed. Let χ_{qk} be the characteristic function of Q_{qk} and let η be an approximative kernel supported on $B(0, 1)$. Let $\eta_q(x) = 2^{3q} \eta(\lambda_q x)$, $\chi'_{qk} = \chi_{qk} \star \eta_q$ and $b_{qk} = u_q \chi'_{qk}$. Then let

$$s_{qk}(t) = \max_{|\beta| \leq M} \{ \lambda_q^{-|\beta|} \|\partial_x^\beta b_{qk}(t, \cdot)\|_\infty \}$$

$$a_{qk} = b_{qk} / s_{qk}.$$

Let us now fix any positive a -dimensional decreasing sequence $\sigma_q \rightarrow 0$ with zero exponential type

$$\lim_{q \rightarrow \infty} \frac{\log \sigma_q}{q} = 0.$$

Definition 3.1. A active region occupied by eddies of size ℓ_p at time t is defined by

$$(23) \quad A_q(t) = \bigcup_{k: |s_{qk}(t)| > \sigma_q \frac{\langle |u_q|^3 \rangle}{\langle |u_q|^2 \rangle}} Q_{qk}^*.$$

Furthermore, let

$$(24) \quad A_q = \{(t, x) : x \in A_q(t)\}.$$

In the following theorem we collect several properties of active regions that are independent of particular nature of the flow or even the evolution law of the field u . For a set $A \subset \Omega_T$, let $A(t) = \{x \in \mathbb{T}_L^3 : (x, t) \in A\}$.

Theorem 3.2. *Let A_q be the active region defined by (23), and let*

$$(25) \quad A = \limsup_{q \rightarrow \infty} A_q = \bigcap_{p=1}^{\infty} \bigcup_{q>p} A_q.$$

Then for some absolute constant $c > 0$ we have

$$(26) \quad |A_q| \leq c \sigma_q^{-3} V_q T,$$

$$(27) \quad (1 - c \sigma_q) \int_{\Omega_T} |u_q|^3 dx dt \leq \int_{A_q} |u|^3 dx dt \leq \int_{\Omega_T} |u_q|^3 dx dt,$$

$$(28) \quad \dim_{\mathcal{H}} A(t) \leq d, \text{ for a.e. } t \in [0, T].$$

In particular, if $d < 0$ then $A(t) = \emptyset$ for a.e. t .

We thus see that the active regions are regions where most of the characteristic velocity U_q is concentrated, yet the measure of A_q on average does not exceed V_q up to an algebraic multiple. Inequality (28) confirms the dimensional prediction of the β -model, and we will argue in the next section that A indeed represents the set where the energy flux accumulates.

Proof. Let $N_q(t)$ be the number of cubes in the union (23). Then trivially

$$|A_q(t)| \leq \frac{27}{\lambda_q^3} N_q(t).$$

On the other hand,

$$\frac{1}{\lambda_q^3} N_q(t) = \left| \bigcup_{k: |s_{qk}(t)| > \sigma_q \frac{\langle |u_q|^3 \rangle}{\langle |u_q|^2 \rangle}} Q_{qk} \right| \leq |A_q(t)|.$$

Integrating in time we obtain

$$(29) \quad \frac{1}{\lambda_q^3} \int_0^T N_q(t) dt \leq |A_q| \leq \frac{27}{\lambda_q^3} \int_0^T N_q(t) dt.$$

In view of (22) we have

$$\int_{\Omega_T} |u|^3 dx dt \sim \frac{1}{\lambda_q^3} \int_0^T \sum_k |s_{qk}(t)|^3 dt \geq \frac{\sigma_q^3 \langle |u_q|^3 \rangle^3}{\lambda_q^3 \langle |u_q|^2 \rangle^3} \int_0^T N_q(t) dt.$$

Thus, in view of (29),

$$|A_q| \leq c \sigma_q^{-3} \frac{\langle |u_q|^2 \rangle^3}{\langle |u_q|^3 \rangle^3} \int_{\Omega_T} |u|^3 dx dt = c \sigma_q^{-3} V_q T,$$

as claimed.

Now let us observe that on the complement $A_q^c = \Omega_T \setminus A$, we have $|u| \leq c \sigma_q \frac{\langle |u_q|^3 \rangle}{\langle |u_q|^2 \rangle}$. We therefore obtain

$$\begin{aligned} \int_{A^c} |u|^3 dx dt &= \int_{A^c} |u|^2 |u| dx dt \leq \sigma_q \frac{\int_{\Omega_T} |u|^3 dx dt}{\int_{\Omega_T} |u|^2 dx dt} \int_{A^c} |u|^2 dx dt \\ &\leq \sigma_q \int_{\Omega_T} |u|^3 dx dt, \end{aligned}$$

which proves (27).

Let us first recall that the Hausdorff dimension of a set $A \subset \mathbb{T}_L^3$, $\dim_{\mathcal{H}}(A)$ is the smallest d for which the Hausdorff measure vanishes

$$\mathcal{H}_{d+\delta}(A) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_{d+\delta, \varepsilon}(A) = 0,$$

for all $\delta > 0$, where

$$\mathcal{H}_{d, \varepsilon}(A) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } A_i)^d : A \subset \cup_i A_i, \text{diam } A_i < \varepsilon \right\}.$$

Directly from (26) we have

$$(30) \quad \frac{1}{T} \int_0^T N_q(t) dt \lesssim \sigma_q^{-3} \lambda_q^3 V_q.$$

Given $\delta > 0$, we note that

$$(31) \quad \frac{V_q}{L^3} \leq 2^{-q(b-\delta/2)},$$

for all q large enough. Let us now assume that $d \geq 0$. Then

$$\mathcal{H}_{d+\delta, \ell_q}(A(t)) \leq \mathcal{H}_{d+\delta, \ell_q}(\cup_{p>q} A_p(t)) \leq \sum_{p>q} N_p(t) \frac{1}{\lambda_p^{d+\delta}}.$$

Integrating in time, using (30), (31) and the fact that $\sigma_q \rightarrow 0$ algebraically, we obtain

$$\int_0^T \mathcal{H}_{d+\delta, \ell_q}(A(t)) dt \leq T \sum_{p>q} \frac{V_p}{\sigma_p^3 \lambda_p^{-b+\delta}} \leq T \sum_{p>q} \frac{L^d 2^{\delta/2}}{\sigma_p^3 \lambda_p^\delta} \rightarrow 0,$$

as $q \rightarrow \infty$. So, in the limit we obtain

$$\int_0^T \mathcal{H}_{d+\delta}(A(t)) dt = 0.$$

Hence, $\dim_{\mathcal{H}} A(t) \leq d + \delta$ for a.e. $t \in [0, T]$, which concludes the proof.

If $d < 0$, then (30) and (31) imply

$$\int_0^T \sum_{p>q} N_p(t) dt \lesssim \sigma_q^{-3} 2^{-q\delta/2},$$

for q large enough. Thus the measure

$$\left| \left\{ t : \sum_{p>q} N_p(t) \geq 1 \right\} \right| \leq \sigma_q^{-3} 2^{-q\delta/2},$$

and on the complement the set $A_q(t)$ is empty. Passing to the limit we conclude that $A(t)$ is empty a.e. \square

4. ENERGY DISSIPATION SET

Let us consider the Euler equations in the periodic box \mathbb{T}_L^3 :

$$(32) \quad \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + f,$$

$$(33) \quad \nabla \cdot u = 0.$$

A vector field $u \in C_w([0, T]; L^2(\mathbb{T}_L^3))$, (the space of weakly continuous functions), is a weak solution of the Euler equations with initial data $u_0 \in L^2(\mathbb{T}_L^3)$ if for every $\psi \in C_0^\infty([0, T] \times \mathbb{T}_L^3)$ with $\nabla_x \cdot \psi = 0$ and $0 \leq t \leq T$, we have

$$(34) \quad \int_{\mathbb{T}_L^3 \times \{t\}} u \cdot \psi - \int_{\mathbb{T}_L^3 \times \{0\}} u_0 \cdot \psi - \int_{\Omega_t} u \cdot \partial_s \psi = \int_{\Omega_t} (u \otimes u) : \nabla \psi + \int_{\Omega_t} f \cdot \psi,$$

and $\nabla_x \cdot u(t) = 0$ in the sense of distributions. We define the operation : by

$$A : B = \text{Tr}[AB].$$

4.1. Energy flux and density. In order to properly define the flux of kinetic energy across the scales, let us fix a $q \geq 1$ and test (32) against the filtered field

$$u_{<q} = \sum_{p=-1}^{q-1} u_p = \mathcal{F}^{-1}(\chi(2^q \cdot / L) \mathcal{F}(u)).$$

Denoting $\tilde{u}_{<q} = \mathcal{F}^{-1}(\sqrt{\chi(2^q \cdot / L)} \mathcal{F}(u))$ which also represents a dyadic filtration, we obtain the following budget relation for the energy dissipation rate across the wavenumber λ_q :

$$(35) \quad \frac{1}{2} \frac{d}{dt} \|\tilde{u}_{<q}\|_2^2 = \int_{\mathbb{T}_L^3} (u \otimes u) : \nabla u_{<q} dx + \int_{\mathbb{T}_L^3} f \cdot u dx,$$

(notice that the latter integral is independent of q for q larger than the integral scale). Let us denote

$$\Pi_q(t) = \int_{\mathbb{T}_L^3} (u \otimes u) : \nabla u_{<q} dx$$

the total energy flux due to nonlinearity. It represents the averaged contribution of all nonlocal interactions to the energy exchange across the wavenumber λ_p . By antisymmetry of the trilinear term, we have

$$\int_{\mathbb{T}_L^3} (u \otimes u_{<q}) : \nabla u_{<q} dx = 0,$$

which is a statement of the fact that eddies larger than ℓ_q on average do not carry the energy across the scale ℓ_q . Thus,

$$(36) \quad \Pi_q(t) = \int_{\mathbb{T}_L^3} (u \otimes u_{\geq q}) : \nabla u_{<q} dx.$$

In this form the flux is clear from the large scale "shuffling". We notice that the Fourier support considerations reduce the formula (36) further to

$$(37) \quad \Pi_q(t) = \int_{\mathbb{T}_L^3} \sum_{\substack{p' \geq q-1, \ p'' \geq q, \ p''' < q \\ |p' - p''| < 2}} (u_{p'} \otimes u_{p''}) : \nabla u_{p'''} dx.$$

This formula shows detailed contribution of scales to the energy budget relation. We will see in Lemma 4.1 that the remote scales do not in fact have a considerable impact on the flux due localization (see [2] and Lemma 4.1 below). We proceed now to define the energy flux density

as the integrand of (37):

$$(38) \quad \pi_q = \sum_{\substack{p' \geq q-1, \ p'' \geq q, \ p''' < q \\ |p' - p''| < 2}} (u_{p'} \otimes u_{p''}) : \nabla u_{p'''}.$$

Thus,

$$(39) \quad \Pi_q(t) = \int_{\mathbb{T}_L^3} \pi_q(x, t) dx.$$

In addition we define for each $K < q$ the truncated density

$$(40) \quad \pi_q^K = \sum_{\substack{p', p'', p''' \text{ as above} \\ p', p'', p''' \in [q-K, q+K]}} (u_{p'} \otimes u_{p''}) : \nabla u_{p'''}.$$

Following the ideas set forward in [2] we obtain a localization property of the flux in the next lemma.

Lemma 4.1. *Suppose that*

$$(41) \quad \varepsilon = \limsup_{q \rightarrow \infty} \lambda_q \langle |u_q|^3 \rangle < \infty.$$

Then we have

$$(42) \quad \limsup_{q \rightarrow \infty} \langle |\pi_q| \rangle \leq c\varepsilon,$$

for some a -dimensional $c > 0$. Moreover,

$$(43) \quad \lim_{K \rightarrow \infty} \limsup_{q \rightarrow \infty} \langle |\pi_q - \pi_q^K| \rangle = 0.$$

Proof. Using the Hölder and differential Bernstein's inequalities, we have

$$\begin{aligned} \langle |\pi_q| \rangle &\leq \sum_{\substack{p' \geq q-1, \ p'' \geq q, \ p''' < q \\ |p' - p''| < 2}} \langle |u_{p'}|^3 \rangle^{1/3} \langle |u_{p''}|^3 \rangle^{1/3} \lambda_{p'''} \langle |u_{p'''}|^3 \rangle^{1/3} \\ &\leq \left(\sum_{p > q-2} \langle |u_p|^3 \rangle \right)^{2/3} \left(\sum_{p < q} \lambda_p^3 \langle |u_p|^3 \rangle \right)^{1/3} \\ &= \left(\sum_{p > q-2} \lambda_q \lambda_p^{-1} \lambda_p \langle |u_p|^3 \rangle \right)^{2/3} \left(\sum_{p < q} \lambda_p^2 \lambda_q^{-2} \lambda_p \langle |u_p|^3 \rangle \right)^{1/3}. \end{aligned}$$

Clearly, the sums above are of convolution type with the exponentially decaying kernels $\lambda_q \lambda_p^{-1}$ and $\lambda_p^2 \lambda_q^{-2}$. Hence, (42) follows. If one

of p', p'', p''' is outside $[q - K, q + K]$, then in at least one of the sums above the summation is performed over p 's with $|p - q| > K$. Thus,

$$\limsup_{q \rightarrow \infty} \langle |\pi_q - \pi_q^K| \rangle \leq c\varepsilon \lambda_K^{-2/3},$$

and (43) follows. \square

4.2. Energy flux and active regions. In this subsection we will connect the energy dissipation with active regions.

Definition 4.2. We say that the energy flux concentrates on a measurable set $G \subset \Omega_T$, or G is an energy dissipation set, if there exists a decreasing sequence of open sets G_q such that $G = \cap_{q=1}^{\infty} G_q$ and

$$(44) \quad \lim_{p \rightarrow \infty} \langle |\pi_p|_{\Omega_T \setminus G_q} \rangle = 0$$

for all $q \in \mathbb{N}$.

We will now prove that the energy flux concentrates on the active region A defined in Theorem 3.2.

Theorem 4.3. Suppose $u \in C_w([0, T]; L^2(\mathbb{T}_L^3))$ is a weak solution to the Euler equations, and

$$(45) \quad \varepsilon = \limsup_{q \rightarrow \infty} \lambda_q \langle |u_q|^3 \rangle < \infty.$$

Then the energy flux concentrates on $A = \limsup_{q \rightarrow \infty} A_q$, where A_q are the active regions.

Proof. Let $G_q = \cup_{p=q}^{\infty} A_p$ and let $G_q^c = \Omega_T \setminus G_q$. According to Lemma 4.1 it suffices to show the limit

$$(46) \quad \lim_{p \rightarrow \infty} \langle |\pi_p^K|_{G_q^c} \rangle = 0,$$

for all $K, q \in \mathbb{N}$. For all $p > q + K$ we obtain the following estimate

$$\begin{aligned} \langle |\pi_p^K|_{G_q^c} \rangle &\leq \sum_{p', p'', p''' \in [p-K, p+K]} \langle |u_{p'}|_{G_q^c} |u_{p''}|_{G_q^c} |\nabla u_{p'''}|_{G_q^c} \rangle \\ &\leq \sum_{p', p'', p''' \in [p-K, p+K]} \langle |u_{p'}|_{A_{p'}^c} |u_{p''}|_{A_{p''}^c} |\nabla u_{p'''}|_{A_{p'''}^c} \rangle. \end{aligned}$$

Let $p^* \in \{p', p'', p'''\}$ be such that

$$\langle |u_{p^*}|^2 \rangle = \max\{\langle |u_{p'}|^2 \rangle, \langle |u_{p''}|^2 \rangle, \langle |u_{p'''}|^2 \rangle\}.$$

We will consider three different cases: $p^* = p'$, $p^* = p''$, and $p^* = p'''$. The first two are similar, so we consider only the first and the last ones.

Assume $p^* = p'$. We have

$$|u_{p'}(t, x)|_{A_{p'}^c} \leq c\sigma_{p'} \frac{\langle |u_{p'}|^3 \rangle}{\langle |u_{p'}|^2 \rangle}.$$

Then we obtain

$$\begin{aligned} \langle |u_{p'}|_{A_{p'}^c} | |u_{p''}|_{A_{p''}^c} | |\nabla u_{p'''}|_{A_{p'''}^c} \rangle &\leq \sigma_{p'} \frac{\langle |u_{p'}|^3 \rangle}{\langle |u_{p'}|^2 \rangle} \langle |u_{p''}| |\nabla u_{p'''}| \rangle \\ &\lesssim \sigma_{p'} \frac{\langle |u_{p'}|^3 \rangle}{\langle |u_{p'}|^2 \rangle} \lambda_{p'''} \langle |u_{p''}|^2 \rangle^{1/2} \langle |u_{p'''}|^2 \rangle^{1/2} \\ &\leq \sigma_{p'} \lambda_{p'''} \langle |u_{p'}|^3 \rangle \\ &\leq \sigma_{p'} \lambda_{2K} \lambda_{p'} \langle |u_{p'}|^3 \rangle \\ &\rightarrow 0, \end{aligned}$$

as $p \rightarrow \infty$.

In the case where $p^* = p'''$ we notice that for every $(t, x) \in A_{p'''}^c$,

$$|\nabla u_{p'''}(t, x)| \leq \left| \sum_k s_{p'''k}(t) \nabla a_{p'''k}(t, x) \right| \lesssim \lambda_{p'''} \sigma_{p'''} \frac{\langle |u_{p'''}|^3 \rangle}{\langle |u_{p'''}|^2 \rangle}.$$

So the argument above works as well. We thus obtain

$$\langle |\pi_p^K|_{G_q^c} \rangle \leq c(K) \varepsilon \sigma_{p-K} \rightarrow 0,$$

as $p \rightarrow \infty$.

□

We note that Theorem 4.3 provides no guarantee that the energy dissipation will occur if $A \neq \emptyset$. In fact at present no example of a weak solution to the Euler equation is known to possess a non-trivial flux $\langle \pi_q \rangle \not\rightarrow 0$, and at the same time to have Onsager regularity $\sup_q \lambda_q \langle |u_q|^3 \rangle < \infty$. In the next section and Section 5 we will find active regions explicitly for some known singular solutions.

4.3. A further confinement of the energy flux. Let us consider the following two dimensional example. Let \vec{e}_j be the vectors of the standard unit basis. We fix a large $s > 0$ and define

$$\begin{aligned} u(x_1, x_2) &= \sum_{q=1}^{\infty} u_q, \\ u_q &= \vec{e}_2 \frac{1}{\lambda_q^s} \sin(\lambda_q x_1). \end{aligned}$$

Clearly u is a smooth stationary parallel shear flow. By a direct computation for this example we have $V_q \sim 1$, so $d = 2$, $s_{qk} \sim \lambda_q^{-s}$ and hence, $A_q \sim \Omega_T$, $A = \Omega_T$. Yet, it is clear that the energy dissipation

set is empty. The reason for this is that the conclusion of Theorem 4.3 is based solely on the intermittency character of the flow, i.e. saturation of the L^3 -average relative to the L^2 -average. This particular example lacks it completely – it is uniformly "active" throughout the flow domain. So, in order to make the prediction of the theorem more precise we need to dispose of the regions where regularity of u exceeds Onsager's threshold, namely $\varepsilon(u) = \limsup_{q \rightarrow \infty} \lambda_q \langle |u_q|^3 \rangle = 0$. In order to state the local version of $\varepsilon(u)$ we use the terminology introduced in [22].

Definition 4.4. Let $u : [0, T] \rightarrow L^2(\mathbb{T}_L^3)$ be a time dependent field. We say that u is (Onsager-)regular if $\varepsilon(u) = 0$. Now let $U \subset \Omega_T$ be a relatively open set. We say that u is regular on U if $(u\varphi)$ is regular for every scalar function $\varphi \in C_0^\infty(U)$. We denote by D the union of all open U 's on which u is regular, and call $S = \Omega_T \setminus D$ the singular set of u . Clearly, S is closed.

As shown in [22] for every weak solution to the Euler equation $u \in C_w([0, T]; L^2(\mathbb{T}_L^3))$ the local energy equality holds on its regular set D :

$$(47) \quad \int_{D(t'')} |u|^2 \varphi - \int_{D(t')} |u|^2 \varphi - \int_D |u|^2 \partial_t \varphi = \int_D (|u|^2 + 2p) u \cdot \nabla \varphi,$$

for all $\varphi \in C_0^\infty(D)$ and $t', t'' \in [0, T]$. In terms of anomalous dissipation distribution introduced by Duchon and Robert [10] it means that $\text{supp } \mathcal{D}(u) \subset S$.

Theorem 4.5. *Under the assumptions of Theorem 4.3 the flux concentrates on $A \cap S$.*

Proof. Let $S_q = \{y \in \Omega_T : \text{dist}\{y, S\} < \frac{1}{q}\}$. Then $S = \cap_q S_q$, $A \cap S = \cap_q (S_q \cap G_q)$, and so, it suffices to show that

$$(48) \quad \lim_{p \rightarrow \infty} \langle |\pi_p^K|_{G_q^c \cup S_q^c} \rangle = 0,$$

for all $K, q \in \mathbb{N}$. Following the proof of Theorem 4.3 we argue

$$\begin{aligned} \langle |\pi_p^K|_{G_q^c \cup S_q^c} \rangle &\leq \sum_{p', p'', p''' \in [p-K, p+K]} \langle |u_{p'}|_{G_q^c} |u_{p''}|_{G_q^c} |\nabla u_{p'''}|_{G_q^c} \rangle \\ &+ \sum_{p', p'', p''' \in [p-K, p+K]} \langle |u_{p'}|_{S_q^c} |u_{p''}|_{S_q^c} |\nabla u_{p'''}|_{S_q^c} \rangle. \end{aligned}$$

It thus suffices to prove convergence of the second sum to zero. Since S_q^c is closed and disjoint from S we can find a cut-off function $\varphi \in C_0^\infty(D)$ such that $\varphi = 1$ on S_q^c . We use the following representation

$$(49) \quad (u\varphi)_p = u_p \varphi + u \varphi_p + r_p(u, \varphi),$$

where

$$r_p(u, \varphi)(x, t) = \int_{\mathbb{T}_L^3} h_p(y)(u(x - y, t) - u(x, t))(\varphi(x - y, t) - \varphi(x, t))dy.$$

Restricting (49) onto S_q^c we obtain

$$u_{p'}|_{S_q^c} = (u\varphi)_{p'}|_{S_q^c} - u\varphi_{p'}|_{S_q^c} - r_{p'}(u, \varphi)|_{S_q^c}.$$

We then trivially estimate

$$\begin{aligned} \langle |u_{p'}|_{S_q^c} |u_{p''}|_{S_q^c} |\nabla u_{p'''}|_{S_q^c} \rangle &\leq \langle |(u\varphi)_{p'}| |u_{p''}| |\nabla u_{p'''}| \rangle \\ &\quad + \langle |u\varphi_{p'}| |u_{p''}| |\nabla u_{p'''}| \rangle \\ &\quad + \langle |r_{p'}(u, \varphi)| |u_{p''}| |\nabla u_{p'''}| \rangle. \end{aligned}$$

For the first term we have

$$\begin{aligned} \langle |(u\varphi)_{p'}| |u_{p''}| |\nabla u_{p'''}| \rangle &\leq C(K)(\lambda_{p'} \langle |(u\varphi)_{p'}|^3 \rangle)^{1/3} (\lambda_{p''} \langle |u_{p''}|^3 \rangle)^{1/3} \times \\ &\quad \times (\lambda_{p'''} \langle |u_{p'''}|^3 \rangle)^{1/3}, \end{aligned}$$

which vanishes as $p \rightarrow \infty$ by the regularity of $u\varphi$. The second term vanishes trivially by the regularity of φ . As to the third, we have by Minkowski's inequality

$$\begin{aligned} \langle |r_{p'}(u, \varphi)|^3 \rangle &\leq \int_{\mathbb{T}_L^3} |h_p(y)| \langle |u(\cdot - y, t) - u(\cdot, t)|^3 |\varphi(\cdot - y, t) - \varphi(\cdot, t)|^3 \rangle dy \\ &\leq c \langle |u|^3 \rangle \|\nabla \varphi\|_\infty \int_{\mathbb{T}_L^3} |h_p(y)| |y| dy \leq C(u, \varphi) \lambda_{p'}^{-1}. \end{aligned}$$

Using this estimate we obtain

$$\begin{aligned} \langle |r_{p'}(u, \varphi)| |u_{p''}| |\nabla u_{p'''}| \rangle &\leq C(u, \varphi) \lambda_{p'}^{-1/3} (\langle |u_{p''}|^3 \rangle)^{1/3} (\lambda_{p'''} \langle |u_{p'''}|^3 \rangle)^{1/3} \\ &\leq C(u, \varphi, K, \varepsilon) \lambda_{p'}^{-1/3} \rightarrow 0. \end{aligned}$$

This concludes the proof of the theorem. \square

4.4. Application to the scaling laws of turbulence. In this section we derive some bounds that replicate the classical power laws of turbulence with intermittency correction in terms of

$$d_q = 3 - \frac{\log_2(L^3/V_q)}{q}.$$

Note that 2^{qd_q} represents the number of active eddies of size l_q . More precisely, thanks to (30) we have

$$\frac{1}{T} \int_0^T N_q(t) dt \lesssim \sigma_q^{-3} (L\lambda_q)^{d_q}.$$

Like before, \lesssim stands for an inequality up to a universal constant.

Let us denote $\bar{\varepsilon} = \sup \lambda_q \langle |u_q|^3 \rangle$, and $\underline{\varepsilon} = \inf \langle |\pi_q| \rangle$. Then

$$\underline{\varepsilon} \leq \langle |\pi_q| \rangle \lesssim \bar{\varepsilon},$$

for all q . Here the second inequality follows from the local estimates on the flux (see [2] or Subsection 4.3):

$$(50) \quad \langle |\pi_q| \rangle \lesssim \sum_p K_{q-p} \lambda_p \|u_p\|_3^3,$$

where K is the following localization kernel

$$(51) \quad K_q = \begin{cases} \lambda_q^{2/3}, & q \leq 0; \\ \lambda_q^{-4/3}, & q > 0. \end{cases}$$

Now let us recall that the energy spectrum in the Littlewood-Paley settings is defined as follows (see [6])

$$(52) \quad E_q = \frac{\langle |u_q|^2 \rangle}{\lambda_q}.$$

To take into account intermittency we notice that

$$(53) \quad V_q = L^{d_q} \lambda_q^{d_q-3}$$

and obtain

$$\langle |u_q|^2 \rangle = \langle |u_q|^3 \rangle^{2/3} V_q^{1/3} L^{-1} = \langle |u_q|^3 \rangle^{2/3} (L \lambda_q)^{d_q/3-1} \leq \frac{L^{d_q/3-1} \bar{\varepsilon}^{2/3}}{\lambda_q^{(5-d_q)/3}}.$$

Hence we recover the energy power law with intermittency correction:

$$(54) \quad E_q \leq \frac{\bar{\varepsilon}^{2/3}}{\lambda_q^{5/3} (L \lambda_q)^{1-d_q/3}}.$$

Note that this bound recovers (1) with $d = \sup d_q$.

Let us now estimate the energy spectrum from below. For this we use the estimates on the flux (50) obtaining

$$\begin{aligned} \underline{\varepsilon}^{2/3} &\leq \langle |\pi_q| \rangle^{2/3} \lesssim \left(\sum_p K_{q-p} \lambda_p \langle |u_p|^3 \rangle \right)^{2/3} \lesssim \sum_p K_{q-p}^{2/3} \lambda_p^{2/3} \langle |u_p|^3 \rangle^{2/3} \\ &= \sum_p K_{q-p}^{2/3} \lambda_p^{2/3} \langle |u_p|^2 \rangle V_p^{1/3} L = \sum_p K_{q-p}^{2/3} \lambda_p^{5/3} (L \lambda_p)^{1-d_p/3} E_p. \end{aligned}$$

Combining with (54) we have the following two-sided estimate:

$$(55) \quad \underline{\varepsilon}^{2/3} \lesssim K^{2/3} * \{ \lambda_q^{5/3} (L \lambda_q)^{1-d_q/3} E_q \}_q \lesssim \bar{\varepsilon}^{2/3}.$$

Notice that since the kernel $K^{2/3}$ has an absolute weight concentrated mostly near 0, the lower bound can be interpreted as the averaged spectrum over nearby wavenumbers. Such an averaging is not uncommon in experimental science.

Next, let us consider the velocity displacement $\delta_y u(x, t) = u(x + y, t) - u(x, t)$ and for $0 < \ell \leq L$ define the generalized isotropic second order structure function by

$$S_2(\ell) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \langle |\delta_{\ell\theta} u|^2 \rangle d\theta.$$

Lemma 4.6. *Let $\bar{\varepsilon} < \infty$ and $d = 3 - \mathfrak{h}$ be defined as before. For every $\delta > 0$ there exists an adimensional $C_\delta > 0$ such that*

$$(56) \quad S_2(\ell) \lesssim \bar{\varepsilon}^{\frac{2}{3}} \ell^{\frac{2}{3}} \left[\left(\frac{\ell}{L} \right)^{1 - \frac{d}{3} - \delta} + C_\delta \left(\frac{\ell}{L} \right)^{\frac{4}{3}} \right],$$

for all $\ell < L$.

Proof. Let us fix $y \neq 0$ and using (53) observe for all $q > 0$

$$(57) \quad \begin{aligned} \langle |\delta_y u|^2 \rangle &\lesssim \sum_{p \leq q} |y|^2 \lambda_p^2 \langle |u_p|^2 \rangle + \sum_{p > q} \langle |u_p|^2 \rangle \\ &\lesssim \bar{\varepsilon}^{2/3} \sum_{p \leq q} |y|^2 \lambda_p^2 V_p^{1/3} \lambda_p^{-2/3} + \bar{\varepsilon}^{2/3} \sum_{p > q} V_p^{1/3} \lambda_p^{-2/3} \\ &\leq \bar{\varepsilon}^{2/3} \sum_{p \leq q} |y|^2 \lambda_p^{4/3} (L\lambda_p)^{\frac{d_p}{3} - 1} + \bar{\varepsilon}^{2/3} \sum_{p > q} (L\lambda_p)^{\frac{d_p}{3} - 1} \lambda_p^{-2/3} \end{aligned}$$

Let $\delta > 0$ be fixed. Then in view of $\limsup_{q \rightarrow \infty} d_q = d$, there exists a $p_0 \in \mathbb{N}$ such that for all $p > p_0$ one has $d_p < d + \delta$. Let us suppose that $|y| < \frac{1}{\lambda_{p_0}}$. Then we choose $q = \lceil \log_2(L|y|^{-1}) \rceil + 1 > p_0$. Notice that $\lambda_q \sim 1/|y|$. Then splitting the first sum in (57) we continue

$$(58) \quad \begin{aligned} \langle |\delta_y u|^2 \rangle &\lesssim C_\delta \bar{\varepsilon}^{2/3} |y|^2 L^{-4/3} + \bar{\varepsilon}^{2/3} |y|^2 \sum_{p_0 < p \leq q} \lambda_p^{4/3} (L\lambda_p)^{d/3 - 1 + \delta} \\ &\quad + \bar{\varepsilon}^{2/3} \sum_{p \geq q} (L\lambda_p)^{d/3 - 1 + \delta} \lambda_p^{-2/3} \\ &\lesssim C_\delta \bar{\varepsilon}^{2/3} |y|^2 L^{-4/3} + \bar{\varepsilon}^{2/3} |y|^2 \lambda_q^{4/3} (L\lambda_q)^{d/3 - 1 + \delta} + \bar{\varepsilon}^{2/3} (L\lambda_q)^{d/3 - 1 + \delta} \lambda_q^{-2/3} \\ &\lesssim C_\delta \bar{\varepsilon}^{2/3} |y|^2 L^{-4/3} + \bar{\varepsilon}^{2/3} (L\lambda_q)^{d/3 - 1 + \delta} \lambda_q^{-2/3} (|y|^2 \lambda_q^2 + 1), \end{aligned}$$

where $C_\delta > 0$ is a constant dependend only on $\delta > 0$. Now recalling that $\lambda_q \sim 1/|y|$, we obtain

$$\langle |\delta_y u|^2 \rangle \lesssim \bar{\varepsilon}^{2/3} |y|^{2/3} \left[C_\delta \left(\frac{|y|}{L} \right)^{4/3} + \left(\frac{|y|}{L} \right)^{1-d/3-\delta} \right],$$

which is (56).

Now let us suppose that $|y| \geq \frac{1}{\lambda_{p_0}}$. Then we choose $q = p_0$ and continue from (57)

$$(59) \quad \langle |\delta_y u|^2 \rangle \lesssim C_\delta \bar{\varepsilon}^{2/3} |y|^2 L^{-4/3} + \bar{\varepsilon}^{2/3} (L \lambda_{p_0})^{\frac{d}{3}-1+\delta} \lambda_{p_0}^{-2/3}$$

Notice that the power of λ_{p_0} is negative, so using our assumption we replace it with $|y|^{-1}$:

$$\begin{aligned} \langle |\delta_y u|^2 \rangle &\lesssim C_\delta \bar{\varepsilon}^{2/3} |y|^2 L^{-4/3} + \bar{\varepsilon}^{2/3} \left(\frac{|y|}{L} \right)^{1-d/3-\delta} |y|^{2/3} \\ &= \bar{\varepsilon}^{2/3} |y|^{2/3} \left[C_\delta \left(\frac{|y|}{L} \right)^{4/3} + \left(\frac{|y|}{L} \right)^{1-d/3-\delta} \right], \end{aligned}$$

which immediately implies (56). \square

By a similar argument using the fact that $\varepsilon < \infty$ we find $\langle |\delta_y u|^3 \rangle \lesssim \varepsilon \ell$. Thus, by interpolation, for any $2 \leq p \leq 3$ we have up to a higher order correction term

$$(60) \quad S_p(\ell) \lesssim \varepsilon^{\frac{p}{3}} \ell^{\zeta_p}, \quad \zeta_p = \frac{p}{3} + (3-d) \left(1 - \frac{p}{3} \right).$$

Formula (60) is exactly the one that appears from the dimensional argument of the β -model. We note however that it is not consistent with experimental results for larger values of p . A finer tuning can be achieved by using the multi-fractal formalism of Frisch and Parisi [21].

4.5. Remarks on multi-fractality: future research. As we have seen, no uni-scaling hypothesis used in the original β -model [14] is necessary to justify some intermittency corrections for deterministic fields. In fact, the accumulant A can be used as a base to build a multi-fractal theory similar to the one proposed in the celebrated work of Frisch and Parisi [21]. In our context this can be achieved by defining a nested sequence of active volumes, regions, dimensions, etc, based on saturation of the L^2 - L^p averages. Indeed, if the argument of Section 2.2 is started by using L^p -averages of the velocity field as characteristic

speeds, one obtains

$$(61) \quad V_q^{(p)} = L^3 \frac{\langle |u_q|^2 \rangle^{\frac{p}{p-2}}}{\langle |u_q|^p \rangle^{\frac{2}{p-2}}}.$$

The threshold speed for an active eddie becomes

$$(62) \quad |s_q^{(p)}| \sim \frac{\langle |u_q|^p \rangle^{\frac{1}{p-2}}}{\langle |u_q|^2 \rangle^{\frac{1}{p-2}}}.$$

One can define active regions $A_q^{(p)}$ and accumulant $A^{(p)}$ in a similar way. By interpolation, one has $A_q^{(p'')} \subset A_q^{(p')}$ for all $p'' > p' \geq 3$, and hence $\{A^{(p)}\}_{3 \leq p \leq \infty}$ defines a foliation of our base set A . Multi-fractality can therefore be viewed as a set of scaling parameters: the dimensions $d^{(p)}$ as co-exponents of the volume sequences $V_q^{(p)}$, and a spectrum of scaling exponents of the field on accumulants $A^{(p)}$.

In a subsequent work the analysis above will be carried out formally, but the basic conclusion remains the same: multi-fractality, just like the other basic concepts introduced in this note, is not an assumption imposed on a field – it is a property of the field determined by its volumetric quantities.

5. EXAMPLES

In this section we examine several examples of fields to illustrate how the active regions can be computed explicitly. We also show that generally there is no inclusion of the carriers A and S in either side.

Let us consider first the stationary vortex sheet solution, where $u(x, y, z) = (H(z), 0, 0)$, where H is the Heaviside function. Then by a direct computation, $\langle |u_q|^p \rangle \sim \ell_q$, for any $p \geq 1$ and $q \geq 1$. Thus, $\varepsilon \in (0, \infty)$, $V_q \sim \ell_q$, and therefore $d = 2$, which is exactly the dimension of the set of discontinuities for u . Notice also that $S = \{z = 0\}$. Furthermore, $A = \{z = 0\}$ as well. One can see it by manufacturing atoms for u out of smoothed characteristic functions of the dyadic cubes. Then s_{qk} become comparable to 1 only about the $1/\lambda_q$ vicinity of S , and zero away from it. This makes A_q be a sequence of $1/\lambda_q$ -thin slabs converging to S . Thus, in the limsup $A_q \rightarrow S$. Note that u is stationary, $f = 0$, and therefore the energy conservation holds, despite of the fact that neither A or S sees it. Even more generally, one can show that for any smooth vortex sheet A and S coincide with the sheet, yet the energy conservation holds. This is due to the particular kinematic condition on the sheet (the particles cannot cross the surface of the sheet) that follows from the weak formulation of the Euler equations (see [22]).

Computation of the active regions in the case of a one-point singularity can be done explicitly too. We recall for a moment a two-dimensional example studied in [23] where u is assumed to be $-1/3$ -homogeneous in the radial direction near the origin, it is a stationary solution to (32) with smooth forcing f , and thus $u = \nabla^\perp \psi$ where $\psi(r, \theta) = r^{2/3} \Psi(\theta)$ near the origin, 0 far from it, and $\Psi \in C^2([0, 2\pi])$. Here (r, θ) stand for the polar coordinates. Clearly $S \subseteq \{0\}$ and it is easy to verify directly by integration that $\langle |u_q|^p \rangle \sim \ell_q^{(2-p/3)}$, for $p \leq 3$. Thus, $S = \{0\}$ and the solution overall is Onsager-critical. Furthermore, $V_q \sim \ell_q^2$, and $d = 2 - \mathfrak{h} = 0$ as it is supposed to be. To find A we note that the condition $|s_{qk}| > \sigma_q \lambda_q^{1/3}$ can only be satisfied near the origin since away from it, $s_{qk} \rightarrow 0$ exponentially fast in q . Thus, $A_q \rightarrow \{0\} = A$. The smoothness of f necessitates the singular part of the pressure to be given by $p = r^{-2/3} P$, P is a constant and Ψ satisfies the following ODE

$$(63) \quad 3(\Psi')^2 + 4\Psi^2 + 6\Psi\Psi'' = P.$$

The flux is given by

$$(64) \quad \Pi = \langle f \cdot u \rangle = \int_0^{2\pi} (\Psi'(\theta))^3 d\theta.$$

As shown in [23], (63) implies $\Pi = 0$, and hence all such stationary solutions have no anomalous dissipation, again in spite of the fact that $A = S \neq \emptyset$. It is not clear at the moment whether there exist at all Onsager-critical weak solutions to the Euler equation with smooth forcing, and a one-point singularity that have non-trivial anomalous flux.

The analogous three dimensional construction can be considered as well, where in spherical coordinates $u = \frac{1}{r^{2/3}} U(\theta, \phi)$ near the origin, and U is smooth vector field. In this case we have $\langle |u_q|^p \rangle \sim \ell_q^{(3-2p/3)}$, for $p \leq 3$, and hence $V_q \sim \ell_q^3$ confirming again the dimension $d = 0$ is consistent with the structure of the singularity. At this moment an examination of the flux as in 2D has not been performed although we have reasons to believe that the result remains the same. We note that the analogue of the ODE (63) becomes now a system of coupled nonlinear second order PDEs on components of U .

So far we have seen examples with the inclusion $S \subseteq A$. Generally, this is not the case. Let us consider cubes of integers C_q of side width

$\lambda_q/10$ placed at the frequency $\lambda_q \vec{e}_1$, and define

$$u = \sum_q u_q,$$

$$\mathcal{F}(u_q)(k) = \vec{e}_2 \frac{1}{\lambda_q^{7/3}} \chi_{C_q}(k).$$

Then $\|u_q\|_3 \sim 1/\lambda_q^{1/3}$, $\|u_q\|_2 \sim 1/\lambda_q^{5/6}$, and hence $V_q \sim 1/\lambda_q^3$, and $d = 0$. The threshold for active eddies becomes $|s_{qk}| > \sigma_q \lambda_q^{2/3}$, yet one can see that u_q is the product of 3 modulated and scaled Dirichlet kernels $\lambda_q^{-7/3} D_{\lambda_q/10}^{\otimes 3}$. Thus $|u_q| \sim \lambda_q^{2/3}$ at the origin and u_q concentrates near the origin as $q \rightarrow 0$. So, by choosing a sequence σ_q decaying slowly enough we can conclude that the eddies over the threshold concentrate near the origin as well. This shows that $A = \{0\}$. Now let $U \subset \mathbb{T}_L^3$ be open and $\varphi \in C_0^\infty(U)$ and $\varphi \geq 0$. Let $\alpha = \int \varphi > 0$. We have $(u\varphi)_q \sim u_q \varphi$ by the same analysis as in the proof of Theorem 4.5. Then due to the smoothness of φ and uniformity of Fourier modes of u_q in the box C_q we obtain

$$(65) \quad \mathcal{F}(u_q \varphi)(k) = \mathcal{F}(u_q) \star \mathcal{F}(\varphi) \sim \alpha / \lambda_q^{7/3}$$

for q large enough and for all k in a sub-box of C_q with edges of size $\lambda_q/10 - N$. The larger the N the more precise the identity (65) becomes in the limit $q \rightarrow \infty$. Thus, up to a negligible error,

$$\|u_q \varphi\|_3 \gtrsim \|\lambda_q^{-7/3} D_{\lambda_q/10-N}^{\otimes 3}\|_3 \gtrsim \alpha / \lambda_q^{1/3}.$$

This shows that $u\varphi$ is Onsager-singular, and thus $S = \mathbb{T}_L^3$.

Finally we note that an example is available with smoothness $1/3$ in the $L^{18/11}$ -average sense ($u \in B_{18/11,\infty}^{1/3}$ in the notation of Besov spaces) that solves the Euler equation with smooth forcing $f = (0, 0, \cos(x))$ and has anomalous dissipation $\langle f \cdot u \rangle \neq 0$. The details of this construction will be presented elsewhere.

6. CONNECTION WITH DUCHON-ROBERT'S APPROACH

Let u be a weak solution to the Euler equation with $\varepsilon < \infty$, and let $\mathcal{D}(u)$ be the distribution of Duchon and Robert as described in the Introduction. From Lemma 4.1 we can see that $\{\pi_q\}_q$ is uniformly bounded in $L^1(\Omega_T)$. By following the same calculations as in [10] one can show that in our terms $\mathcal{D}(u)$ is the weak*-limit of this sequence, hence $\mathcal{D}(u)$ is a measure of bounded variation. In this section we will use the active regions to describe a measure-theoretic support of $\mathcal{D}(u)$ in the sense of Hahn decomposition.

We first make several definitions. Let $B \subset \Omega_T$ be a measurable set, and $k \geq 0$ be an integer. We say that B is a k -cluster of the sequence $\{A_q\}_q$ if for every open set U containing B one has

$$(66) \quad \lim_{Q \rightarrow \infty} \max \{ |q_1 - q_2| : q_i > Q, (\cup_{q_1 \leq q \leq q_2} A_q) \cap U = \emptyset \} = k.$$

In other words for very U the gap in the string of indices q such that $A_q \cap U = \emptyset$ eventually does not exceed k , and k is the minimal such gap. If $k = \infty$, we define a ∞ -cluster as a set B which is not a k -cluster for any finite $k \geq 0$.

Let B be a closed k -cluster. Denote $k = k(B)$. Clearly, if $B_1 \subset B_2$, then $k(B_1) \geq k(B_2)$. We call B minimal if for every proper closed subset B' of B , one has $k(B') > k(B)$. By Zorn's lemma every closed k -cluster set B , with finite k , contains a minimal k -cluster subset. Indeed, let $\{B_\alpha\}_{\alpha \in I}$ be a chain of k -clusters ordered by inclusion. Let $B = \cap_{\alpha \in I} B_\alpha$. Then for every U containing B , $B_\alpha \subset U$ eventually, i.e. for all $\alpha > \alpha_0$. Since B_α is a k -cluster, (66) holds. This shows that B is a lower bound for the chain, and Zorn's lemma applies.

Consider the set

$$M = \bigcup_{0 \leq k < \infty} \bigcup_{B \text{ is minimal } k\text{-cluster}} B.$$

Theorem 6.1. *Suppose u is a weak solutions to the Euler equation with $\varepsilon < \infty$. Then*

- (i) $\Omega_T \setminus M$ is a null-set of $\mathcal{D}(u)$;
- (ii) $\text{supp } \mathcal{D}(u) \subset \overline{M} \subset \cap_{q \geq 1} \overline{\cup_{p > q} A_p}$.

Proof. For (i) it is enough to show that $\mathcal{D}(F) = 0$ for every closed subset of $\Omega_T \setminus M$. Since $F \cap B = \emptyset$ for any minimal k -cluster set B , and F is closed, F itself is not a k -cluster for any finite k . Hence, for every $k > 0$ there exists an open $U \supset F$ and for any $N > 0$ there exists a string $q'_N, \dots, q''_N > N$ with $|q'_N - q''_N| > 2k + 1$ such that $A_q \cap U = \emptyset$, for all $q'_N \leq q \leq q''_N$. Let $q_N = [(q'_N + q''_N)/2]$ and let $\phi \in C_0^\infty(U)$ be arbitrary. Then

$$\int_{\Omega_T} \phi \pi_{q_N} dx dt \rightarrow \int_{\Omega_T} \phi d\mathcal{D},$$

as $N \rightarrow \infty$. On the other hand, by the same computations as in the proof of Theorem 4.3 and since there are no active regions in U with indices near q_N , we obtain

$$\left| \int_{\Omega_T} \pi_{q_N}^k \phi dx dt \right| \leq C \sigma_{q'_N} \rightarrow 0,$$

as $N \rightarrow \infty$. Thus, in view of Lemma 4.1,

$$\left| \int_{\Omega_T} \phi d\mathcal{D} \right| \leq o(1) \|\phi\|_\infty,$$

where $o(1) \rightarrow 0$ as $k \rightarrow \infty$. This implies $|\mathcal{D}|(F) \leq |\mathcal{D}|(U) \leq o(1)$. Since F is independent of k , letting $k \rightarrow \infty$ we obtain the desired result.

The conclusions of (ii) are even more straightforward as seen from the argument above – there is no clustering of active regions at all outside of \overline{M} .

□

REFERENCES

- [1] David R. Adams and Lars Inge Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.
- [2] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager’s conjecture for the Euler equations. *Nonlinearity*, 21(6):1233–1252, 2008.
- [3] A. Cheskidov and R. Shvydkoy. A unified approach to regularity problems for the 3d navier-stokes and euler equations: the use of kolmogorov’s dissipation range. <http://arxiv.com/abs/1102.1944>.
- [4] Alexey Cheskidov and Susan Friedlander. The vanishing viscosity limit for a dyadic model. *Phys. D*, 238(8):783–787, 2009.
- [5] Alexey Cheskidov, Susan Friedlander, and Roman Shvydkoy. A continuous model for turbulent energy cascade. arXiv:1112.5376v1.
- [6] P. Constantin. The littlewood-paley spectrum in 2d turbulence. *Theor. Comp. Fluid Dyn.*, 9(2):183–189, 1997.
- [7] P. Constantin, Q. Nie, and S. Tanveer. Bounds for second order structure functions and energy spectrum in turbulence. *Phys. Fluids*, 11(8):2251–2256, 1999. The International Conference on Turbulence (Los Alamos, NM, 1998).
- [8] Peter Constantin, Weinan E, and Edriss S. Titi. Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. *Comm. Math. Phys.*, 165(1):207–209, 1994.
- [9] Peter Constantin and Charles Fefferman. Scaling exponents in fluid turbulence: some analytic results. *Nonlinearity*, 7(1):41–57, 1994.
- [10] Jean Duchon and Raoul Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity*, 13(1):249–255, 2000.
- [11] Gregory L. Eyink. Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. *Phys. D*, 78(3-4):222–240, 1994.
- [12] Gregory L. Eyink. Besov spaces and the multifractal hypothesis. *J. Statist. Phys.*, 78(1-2):353–375, 1995. Papers dedicated to the memory of Lars Onsager.
- [13] Ciprian Foias. What do the Navier-Stokes equations tell us about turbulence? In *Harmonic analysis and nonlinear differential equations (Riverside, CA,*

- 1995), volume 208 of *Contemp. Math.*, pages 151–180. Amer. Math. Soc., Providence, RI, 1997.
- [14] U. Frisch, P.-L. Sulem, and M. Nelkin. A simple dynamical model of intermittent fully developed turbulence. *J. Fluid Mech.*, 87:719–736, 1978.
 - [15] Uriel Frisch. *Turbulence*. Cambridge University Press, Cambridge, 1995. The legacy of A. N. Kolmogorov.
 - [16] Benoit Mandelbrot. Intermittent turbulence and fractal dimension: kurtosis and the spectral exponent $5/3 + B$. In *Turbulence and Navier-Stokes equations (Proc. Conf., Univ. Paris-Sud, Orsay, 1975)*, pages 121–145. Lecture Notes in Math., Vol. 565. Springer, Berlin, 1976.
 - [17] W. D. McComb. *The physics of fluid turbulence*, volume 25 of *Oxford Engineering Science Series*. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.
 - [18] A. S. Monin and A. M. Yaglom. *Statistical fluid mechanics: mechanics of turbulence. Vol. II*. Dover Publications Inc., Mineola, NY, 2007. Translated from the 1965 Russian original, Edited and with a preface by John L. Lumley, English edition updated, augmented and revised by the authors, Reprinted from the 1975 edition.
 - [19] E. A. Novikov and R. W. Stewart. The intermittency of turbulence and the spectrum of energy dissipation. *Izv. Akad. Nauk SSSR, Ser. Geofiz.*, pages 408–413, 1964.
 - [20] L. Onsager. Statistical hydrodynamics. *Nuovo Cimento (9)*, 6(Supplemento, 2(Convegno Internazionale di Meccanica Statistica)):279–287, 1949.
 - [21] G. Parisi and U. Frisch. On the singularity structure of fully developed turbulence. In G. Parisi M. Ghil, R. Benzi, editor, *Turbulence and Predictability in Geophysical Fluid Dynamics. Proceed. Intern.School of Phys. 'E. Fermi'*, pages 84–87. North-Holland, Amsterdam, Varenna, Italy, 1983.
 - [22] Roman Shvydkoy. On the energy of inviscid singular flows. *J. Math. Anal. Appl.*, 349(2):583–595, 2009.
 - [23] Roman Shvydkoy. Lectures on the Onsager conjecture. *Discrete Contin. Dyn. Syst. Ser. S*, 3(3):473–496, 2010.

(A. Cheskidov) DEPARTMENT OF MATHEMATICS, STAT. AND COMP. SCI.,
UNIVERSITY OF ILLINOIS, CHICAGO, IL 60607

E-mail address: acheskid@math.uic.edu

(R. Shvydkoy) DEPARTMENT OF MATHEMATICS, STAT. AND COMP. SCI.,
UNIVERSITY OF ILLINOIS, CHICAGO, IL 60607

E-mail address: shvydkoy@math.uic.edu